23. W. C. Reynolds and A. K. M. F. Hussain, "Mechanics of an organized wave in turbulent shear flow. Part 3. Theoretical models and comparisons with experiments," J. Fluid Mech., 54, No. 2 (1972).
24. R. A. Antonia, L. W. B. Browne, and L. Fulachier, "Average wavelength of organized structures in the turbulent far wake of a cylinder," Exp. Fluids, 5, No. 5 (1987).
25. M. Lessen and P. J. Sing, "Stability of axisymmetric free shear layers," J. Fluid Mech., 60, No. 3 (1973).
26. M. A. Gol'dshtik and V. N. Shtern, Hydrodynamic Stability and Turbulence [in Russian], Nauka, Novosibirsk (1977).

EVOLUTION AND INTERACTION OF THREE-DIMENSIONAL VORTEX CLUSTERS

G. A. Kuz'min UDC 532.517.4

TURBULENT FLOW MODEL: ENSEMBLE OF SMALL VORTICES

Hydrodynamic instabilities in turbulent flows lead to the formation of concentrated pockets of vorticity (clusters). Their evolution in time is governed by the nonlinear vorticity dynamics in the interior of the vortices and by their mutual interaction.

The possibility of analyzing separately the internal and external degrees of freedom depends on the intermittency factor $\varkappa = \lambda/\ell$ (λ is a characteristic length of the vortices, and ℓ is the distance between them). If $\varkappa \to 0$, the vortices interact only through their momenta, and the other degrees of freedom are insignificant [1, 2].

If $\varkappa \neq 0$, other multipole moments take part in the interaction of the vortices. In turn, their evolution is determined not only by the effect of the surroundings on each specific vortex, but also by the nonlinear dynamics of all internal degrees of freedom, the set of which is not exhausted by the multipole moments [2, 3].

The influence of the vortex surroundings on its internal degrees of freedom for $\varkappa \ll 1$ is similar to the influence of a certain nonuniform external velocity field. Consequently, the total system of equations for the ensemble of small vortex clusters is partitioned into subsystems. Each subsystem describes a particular vortex in the external field induced by the other vortices. The objective of the present study is to derive such a subsystem of equations and to analyze its solutions.

VORTEX CLUSTER IN AN EXTERNAL FIELD IN AN INFINITE COMPRESSIBLE FLUID

The vorticity field obeys the equation

$$\partial\boldsymbol{\omega}/\partial t - \nu\Delta\boldsymbol{\omega} = (\boldsymbol{\omega}\nabla)\mathbf{u} - (\mathbf{u}\nabla)\boldsymbol{\omega}. \quad (1)$$

Galerkin's method is used for the approximate solution of Eq. (1). The choice of basis for the expansion is based on the following considerations.

Vortex clusters in turbulent flows comprise certain irregular diffuse formations. If the Reynolds number Re determined from the cluster parameters is small, the evolution of a vortex depends mainly on the viscosity. Consequently, a natural basis for the expansion is the set of solutions of the linearized equation (1).

Turbulent fluctuations having a broad spectrum of space scales develop inside the vortices for large Re. The detailed description of these fluctuations would require the inclusion of a large number of terms in the expansion, regardless of the system of functions chosen as the basis. Large-scale vortex deformations, which influence the interaction between the vortices, are the most important in regard to the present study. Small-scale fluctuations act as a reservoir, from which energy is drained. Their influence can be taken into account by means of an effective viscosity coefficient ν_{ef}. The number Re_{ef} formulated using the effective viscostiy is no longer as large as Re, and the solution of the linearized equation (1) with ν replaced by ν_{ef} can be adopted as the basis of the expansion.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 44-48, March-April, 1991. Original article submitted March 9, 1989; revision submitted October 2, 1989.

 0021-8944/91/3202-0184$12.50

The multiple moments of the vortices and such important physical characteristics as the vortex momentum and angular momentum are related to the coefficients of the expansion with respect to the basis by simple equations. This fact offers an additional argument in favor of the selected basis system of functions.

Certain particular solutions of the linearized equation (1) (in a special coordinate system) have been investigated previously [4]. These solutions are expressed in terms of orthogonal Hermite polynomials. The general solution of the linearized equations (1) are expressed in covariant form in terms of tensor Hermite polynomials, which are given by the equations [5]

$$H^{(n)}_{i_1\ldots i_n}\left(\frac{\mathbf{r}}{\lambda}\right)=(-\lambda)^n\exp\left(\frac{r^2}{\lambda^2}\right)\frac{\partial^n\exp\left(-\frac{r^2}{\lambda^2}\right)}{\partial r_{i_1}\ldots\partial r_{i_n}}. \tag{2}$$

The polynomials (2) satisfy the orthogonality relations

$$\frac{1}{\pi^{3/2}\lambda^3}\int H^{(n)}_{i_1\ldots i_n}\left(\frac{\mathbf{r}}{\lambda}\right)H^{(m)}_{j_1\ldots j_m}\left(\frac{\mathbf{r}}{\lambda}\right)\exp\left(\frac{-r^2}{\lambda^2}\right)dV=2^n\delta_{nm}\Delta^{j_1\ldots j_n}_{i_1\ldots i_n}, \tag{3}$$

where $\Delta^{j_1\ldots j_n}_{i_1\ldots i_n}$ is the sum of all possible products of Kronecker delta symbols of the form

$$\Delta^j_i=\delta_{ij},\quad \Delta^{jm}_{il}=\delta_{ij}\delta_{lm}+\delta_{im}\delta_{lj}\ldots\ .$$

The general solution of Eq. (1) is sought in the form of an expansion

$$\omega_i(\zeta+\mathbf{r},t)=\frac{\exp\left(-\frac{r^2}{\lambda^2}\right)}{\pi^{3/2}\lambda^3}\sum_{n=1}^{\infty}\frac{1}{\lambda^n}c^i_{i_1\ldots i_n}(\zeta,t)\,H^{(n)}_{i_1\ldots i_n}\left(\frac{\mathbf{r}}{\lambda}\right) \tag{4}$$

[ζ denotes the coordinates of the center of the vortex, and $\lambda(t)$ is its scale]. The orthogonality relations (3) can be used to express the tensors c in terms of the vorticity moments:

$$c^i_{i_1\ldots i_n}(\zeta,t)=\frac{\lambda^n}{2^n n!}\int\omega_i(\zeta+\mathbf{r},t)\,H^{(n)}_{i_1\ldots i_n}\left(\frac{\mathbf{r}}{\lambda}\right)dV. \tag{5}$$

The condition $\operatorname{div}\omega=0$ gives

$$c^i_{i_1\ldots i_n}+c^{i_1}_{i,i_2\ldots i_n}+\ldots+c^{i_n}_{i_1\ldots i_{n-1}\cdot i}=0. \tag{6}$$

The approximation used in the present study stipulates that only a certain finite number of terms N is included explicitly in the sum (4). If there is strong small-scale motion described by terms of the sum (4) with n > N, it is taken into account implicitly through the effective viscosity coefficient. The effective viscosity is estimated by comparison with experiment. This technique is known in the literature on computational fluid dynamics as modeling of the motions of partial grid scales.

We substitute the expansion (4) in Eq. (1) and project the basis onto the first N terms. We then solve the resulting system of ordinary differential equations. The vortex cluster is analyzed in an accompanying frame, in which ζ = 0. For N = 3 the system of equations for the moments has the form

$$\frac{dp_i}{dt}=e_{ijm}\sum_{n=1}^{3}c^m_{i_1\ldots i_n}\frac{\partial^n U_j}{\partial\zeta_{i_1}\ldots\partial\zeta_{i_n}}\bigg|_{\zeta=0}; \tag{7}$$

$$e_{ijm}\frac{dc^m_{kj}}{dt}=\frac{3}{2}\left(b_{ik}-\frac{1}{3}\delta_{ik}b_{jj}\right); \tag{8}$$

$$e_{ijm}\frac{dc^m_{klj}}{dt}+\frac{2}{3}\left(\lambda\frac{d\lambda}{dt}-2\nu\right)\left(p_i\delta_{kl}-\frac{1}{4}p_k\delta_{il}-\frac{1}{4}p_l\delta_{ik}\right)=\frac{\lambda^2}{6}\left(B^i_{kl}-\frac{1}{4}B^{\nu}_{\nu k}\delta_{il}-\frac{1}{4}B^{\nu}_{\nu l}\delta_{ik}\right), \tag{9}$$

where U is the external velocity field, $p_i=e_{ijm}c^m_j$ is the vortex momentum, and b_{ik} and $B^i_{k\ell}$ are quadratic forms in the moments:

$$b_{ik}=e_{ijl}\sum_{n=1}^{3}c^l_{i_1\ldots i_n}\left(\frac{\lambda^2}{2}\frac{\partial^{n+1}U_j}{\partial\zeta_{i_1}\ldots\partial\zeta_{i_n}\partial\zeta_k}+n\delta_{i_n k}\frac{\partial^{n-1}U_j}{\partial\zeta_{i_1}\ldots\partial\zeta_{i_{n-1}}}\right)\bigg|_{\zeta=0}+$$

$$
\begin{aligned}
&+\frac{1}{(2\pi)^{3/2}}\Big[-\frac{7}{15\lambda^3}c_i^m c_k^m+\frac{1}{35\lambda^5}\big(6c_i^m c_{\nu\nu k}^m+6c_k^m c_{\nu\nu i}^m+12c_\nu^m c_{\nu ik}^m+21c_m^i c_{\nu\nu m}^k+\\
&\quad+21c_m^k c_{\nu\nu m}^i-4c_{ik}^m c_{\nu\nu}^m-8c_{\nu i}^m c_{\nu k}^m-7c_{\mu\mu}^i c_{\nu\nu}^k-14c_{\mu\nu}^i c_{\mu\nu}^k\big)-\\
&\quad-\frac{1}{7\lambda^7}\big(2c_{i\nu\nu}^m c_{k\mu\mu}^m+9c_{\mu\mu m}^i c_{\nu\nu m}^k+4c_{ik\nu}^m c_{\mu\mu\nu}^m+4c_{i\mu\nu}^m c_{k\mu\nu}^m+6c_{\mu\nu\sigma}^i c_{\mu\nu\sigma}^k\big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
B_{kl}^i=e_{ijm}\Bigg[&c_\mu^m\left(\lambda^2\frac{\partial^3U_j}{\partial\zeta_k\,\partial\zeta_l\,\partial\zeta_\mu}+2\delta_{\mu l}\frac{\partial U_j}{\partial\zeta_k}+2\delta_{\mu k}\frac{\partial U_j}{\partial\zeta_l}\right)+\\
&+c_{\mu\nu}^m\left(\lambda^2\frac{\partial^4U_j}{\partial\zeta_\mu\,\partial\zeta_\nu\,\partial\zeta_k\,\partial\zeta_l}+4\delta_{\nu l}\frac{\partial^2U_j}{\partial\zeta_\mu\,\partial\zeta_k}+4\delta_{\nu k}\frac{\partial^2U_j}{\partial\zeta_\mu\,\partial\zeta_l}+\frac{8}{\lambda^2}\delta_{\nu k}\delta_{\mu l}U_j\right)+\\
&+c_{\mu\nu\sigma}^m\left(\lambda^2\frac{\partial^5U_j}{\partial\zeta_\mu\,\partial\zeta_\nu\,\partial\zeta_\sigma\,\partial\zeta_k\,\partial\zeta_l}+6\delta_{\mu k}\frac{\partial^3U_j}{\partial\zeta_\nu\,\partial\zeta_\sigma\,\partial\zeta_l}+6\delta_{\mu l}\frac{\partial^3U_j}{\partial\zeta_\nu\,\partial\zeta_\sigma\,\partial\zeta_k}+\right.\\
&\left.+\frac{24}{\lambda^2}\delta_{\mu k}\delta_{\nu l}\frac{\partial U_j}{\partial\zeta_\sigma}\right)\Bigg]\Bigg|_{\zeta=0}+\frac{2}{105\,(2\pi)^{3/2}}\Big\{\frac{1}{\lambda^5}\big[42\left(c_k^m c_{lm}^i+c_l^m c_{km}^i\right)+154c_m^i c_{kl}^m+\\
&+21\left(c_k^i c_{\nu\nu}^l+c_l^i c_{\nu\nu}^k\right)\big]+\frac{1}{\lambda^7}\big[-45\left(c_{\nu\nu}^k c_{\mu\mu l}^i+c_{\nu\nu}^l c_{\mu\mu k}^i\right)-162c_{kl}^m c_{\nu\nu m}^i+\\
&+72c_{kl}^m c_{i\nu\nu}^m+180\left(c_{\mu k}^m c_{\mu ml}^i+c_{\mu l}^m c_{\mu mk}^i\right)-72c_{\nu\nu}^m c_{ikl}^m-81\left(c_{\nu\nu}^i c_{k\mu\mu}^l+c_{\nu\nu}^i c_{l\mu\mu}^k\right)+\\
&+324c_{\nu m}^i c_{\nu kl}^m-144c_{i\nu}^m c_{\nu kl}^m+90\left(c_{km}^i c_{\nu\nu m}^l+c_{lm}^i c_{\nu\nu m}^k\right)\big]\Big\}.
\end{aligned}
$$

The system (7)-(9) consists of 39 scalar equations. By virtue of Eq. (6) and the symmetry conditions, only 26 of the equations are independent. The properties of the solutions of the system of equations are conveniently illustrated in a simple special example, in which the number of independent equations is rendered small by additional symmetry properties of the vortex.

SYMMETRIC VORTEX OSCILLATIONS

Let us assume that the vortex lines are invariant under reflections in the xz and yz planes:

$$
\begin{aligned}
&\omega_x(x,\ y,\ z)=\omega_x(-x,\ y,\ z)=-\omega_x(x,\ -y,\ z),\\
&\omega_y(x,\ y,\ z)=-\omega_y(-x,\ y,\ z)=\omega_y(x,\ -y,\ z),\\
&\omega_z(x,\ y,\ z)=-\omega_z(-x,\ y,\ z)=-\omega_z(x,\ -y,\ z).
\end{aligned}
\tag{10}
$$

Using Eqs. (5), (6), and (10), we express the nonzero components of the tensor moments in terms of six independent functions ε, c, k, d, ℓ, f:

$$
\begin{aligned}
&c_{yz}^x=\varepsilon,\ c_{xz}^y=c,\ c_{xy}^z=-\varepsilon-c,\ c_{xxy}^x=-k,\ c_{yzz}^x=-l,\ c_{yyy}^x=-3d,\\
&c_{xxx}^y=3k,\ c_{xyy}^y=d,\ c_{xzz}^y=f,\ c_{xyz}^z=(l-f)/2.
\end{aligned}
\tag{11}
$$

If conditions (10) hold, the vortex moment $\mathbf{p}$ is directed along the z axis, and the vortex angular momentum is equal to zero.

To close the system of equations, it is necessary to relate the vortex position and scale to the moments of the vorticity distribution. We use the definitions for ζ [6] and λ:

$$
\zeta_z=\frac{1}{2p}\int[\mathbf{r}\times\boldsymbol{\omega}]_z\,z\,dV;
\tag{12}
$$

$$
\lambda^2=\frac{1}{4p}\int[\mathbf{r}\times\boldsymbol{\omega}]_z\,(x^2+y^2)\,dV.
\tag{12'}
$$

If the origin is placed at the center of vorticity, $\zeta = 0$. Relation (12)' is equivalent to $k + d = 0$. If the functions (11) and (12) are known, the vorticity field is reconstructed according to Eq. (4).

The significance of the parameters (11) is determined by the form of the kernels in Eq. (5). The parameter $\ell + f$ gives the vortex length in the longitudinal direction. If $k - d \neq 0$, the vortex is flattened either in the x-direction or in the y-direction, depending on the sign of $k - d$. For $\ell - f \neq 0$ the vortex resembles either an ellipse or a butterfly in the intersection with the xz and yz planes. If, in addition, $c \neq 0$, then the ellipse curves into a banana figure. All these configurations supersede one another in a definite sequence in unsteady vortices.

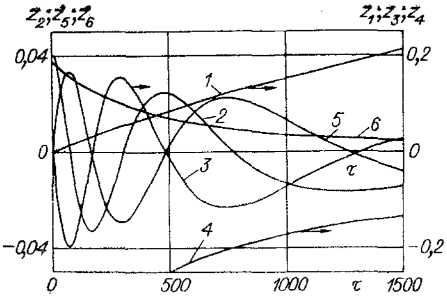

Fig. 1

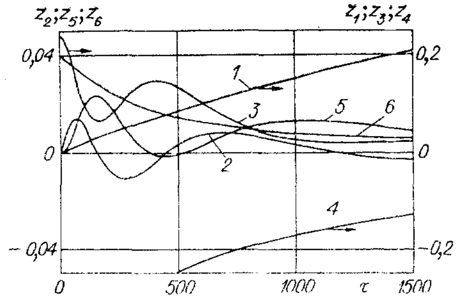

Fig. 2

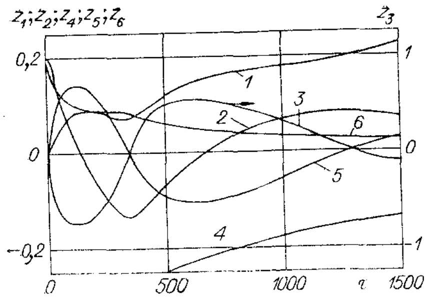

Fig. 3

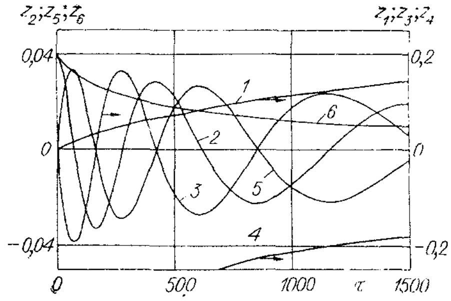

Fig. 4

STEADY-STATE SOLUTIONS

In an inviscid fluid we have an analytical steady-state solution, in which all the vortex lines are rings: $k = d = c = \varepsilon = 0$, $\ell = \text{const}$, $f = \text{const}$, $\lambda = \lambda_0$. The vortex velocity is determined by its momentum, radius, and longitudinal scale.

A vortex similar to a spherical Hill vortex is obtained if Eqs. (11) and (12) are set equal to the corresponding moments of the Hill vortex [1]. The direct computation of the integrals gives $\varepsilon = c = k = d = \ell = f = 0$, $\lambda^2 = (2/7)a^2$ (a is the radius of the Hill vortex). The vortex moves with a velocity $7^{5/2}p/[15(4\pi)^{3/2}a^3]$, which is 20% higher than the velocity of the Hill vortex.

Other steady-state solutions similar to curved vortex rings are readily obtained. We shall not write out these solutions, because it is not clear which of them are preserved in the limit of the infinitely precise approximation. The steady-state solutions can include configurations, which are like vortons of two-dimensional hydrodynamics [7] in that they move through space without changing their shape.

VORTEX OSCILLATIONS

The nature of the evolution of a vortex in a viscous fluid depends on the value of $R = p/\lambda_0^2\nu$. The initial deformations decay monotonically for $R < {\sim}10^3$.

Figures 1-4 show that the equations have oscillating solutions for sufficiently large R. Curves 1-6 correspond to the dimensionless functions

$$z_1 = (\lambda/\lambda_0 - 1)/4,\ z_2 = c/(\lambda p),\ z_3 = 12k/(\lambda^2 p),$$
$$z_4 = 3(l + f)/(2\lambda^2 p),\ z_5 = 3(l - f)/(2\lambda^2 p),\ z_6 = -U\lambda^3/p$$

(U is the self-induced vortex velocity). The external velocity is assumed to be equal to zero. The dimensionless time $\tau = tp/\lambda_0^4$ is plotted along the horizontal axis. In Figs. 1-3 R = 2500.

If a small quadrupole moment z_2 is given at the initial time, curves 1-6 are close to the analytic solution, expressed in terms of Bessel functions, of the linearized (with respect to the deformation amplitude) system of equations (see Fig. 1). In an inviscid fluid z_2, z_3, and z_5 are sinusoidal, and z_1, z_4, and z_6 are constant.

Figure 2 shows the numerical solution of the equations when the vortex lines are elliptical at the initial time [$z_3(0)$ is small, but not equal to zero]. The entire oscillation patern is shifted along the vertical axis. In this regard, diffuse vortices behave differently from thin vortex rings, for which the oscillations do not shift along the vertical axis [8].

Figure 3 illustrates the influence of nonlinearity on the period and waveform of the oscillations. As the amplitude increases, the period of the oscillations increases. The peaks of the function z_2 begin to resemble sawteeth. The peaks of the functions z_3 and z_5, on the other hand, are rounded near the top.

Figure 4 shows the damped oscillations of a turbulent ring. The influence of small-scale turbulence is taken into account through the turbulent viscosity coefficient, which is presumed equal to bp/λ^2 on the basis of dimensional considerations. A comparison of the calculated values of the rate of growth of the vortex scale and translational velocity with the experimental data [9, 10] enables us to estimate the empirical constant b. In Fig. 4 we have $b = 4 \cdot 10^{-4}$ and $R = 10^6$. Consequently, the Reynolds number determined from the turbulent viscosity is quite large, so that the deformations of the turbulent vortex are damped in an oscillatory regime.

INTERACTING VORTICES

The system of equations for an ensemble of vortices consists of subsystems. Each subsystem has the form (7)-(9) and describes the evolution of the parameters and coordinates of one of the vortices. Interaction takes place through the nouniform velocity induced by the vortex system at the site of the given vortex. This velocity is given by a multiple expansion, in which higher than third-order terms are discarded. Roberts' equations [2] are obtained from Eqs. (7)-(9) when the viscosity and the intermittency parameter tend to zero.

LITERATURE CITED

1. G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge Univ. Press (1973).
2. P. H. Roberts, "A Hamiltonian theory for weakly interacting vortices," Mathematika, 19, No. 1 (1970).
3. G. A. Kuz'min and A. Z. Patashinskii, "Tensor characteristics of coherent vortex structures," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1986).
4. A. Sattar, "About transverse vortex rings that can be used as elements of turbulence," Bull. Classe Sci. Acad. Roy. Belg., 67, No. 7 (1984).
5. G. Beitman and A. Erdelyi, Higher Transcendental Functions (California Institute of Technology H. Bateman Manuscript Project), 3 vols., McGraw-Hill, New York (1953, 1955).
6. P. G. Saffman, "The velocity of viscous vortex rings," Stud. Appl. Math., 49, No. 1 (1970).
7. G. S. Deem and N. J. Zabusky, "Vortex waves: stationary 'v states,' interactions, recurrence, and breaking," Phys. Rev. Lett., 40, No. 13, 859-862 (1978).
8. H. Viets and P. M. Sforza, "Dynamics of bilaterally symmetric vortex rings," Phys. Fluids, 15, No. 2 (1972).
9. B. A. Lugovtsov, "Motion of a turbulent vortex ring and the transport of a passive impurity by it," in: Selected Problems in Mathematics and Mechanics [in Russian], Nauka, Leningrad (1970).
10. T. Maxworthy, "Turbulent vortex rings," J. Fluid Mech., 64, No. 2 (1974).